On the Generalization of Butterfly Structure

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FSE 2018, Bruges, Belgium

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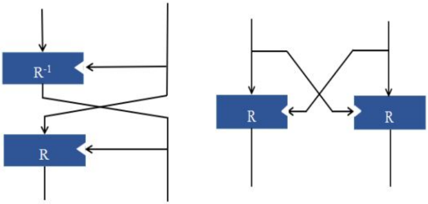
Outlines

1. Backgroud
2. Our generalization and main result
3. Proofs
4. Comparison
5. Future work

Butterfly Structure

A structure that serves infinite family of permutations over $\mathbb{F}_{2^{2n}}$.



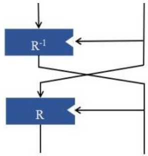
The open butterfly H_R The closed butterfly V_R

- $R_y : x \mapsto R(x, y)$ is a permutation over $\mathbb{F}_{2^n}$ for all y in $\mathbb{F}_{2^n}$;
- $\mathrm{H_R}$ is an involution;
- $\mathrm{H_R}$ and $\mathrm{V_R}$ are CCZ-equivalent;

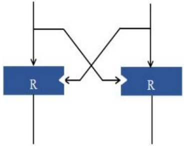
Origin

Crypto 2016, Perrin et.al. reverse-engineered the only known APN permutation over $\mathbb{F}_{2^6}$ and discover this structure.

$$R(x,y) = (x + \alpha y)^3 + y^3$$



The open butterfly H_R



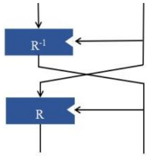
The closed butterfly V_R

- $\alpha \neq 0, 1$ and odd n:
  - Differential uniformity: at most 4;
  - Algebraic degree: $n+1$ or n for H_R and 2 for V_R;
  - Non-linearity: $2^{2n-1} - 2^n$?
- $\alpha = 1$ and odd n: $H_R \Leftrightarrow F^e$ (3-round Feistel) with
  - Differential spectrum: $\{0, 4\}$;
  - Non-linearity: $2^{2n-1} - 2^n$;
  - Algebraic degree: n for H_R and 2 for V_R;

$n > 3$, more APN permutations from H_R?

Previous generalizations

TIT 2017, Anne Canteaut et. al.: $(x+\alpha y)^3 + y^3 \Rightarrow \ (x+\alpha y)^3 + \beta y^3$ with $\alpha, \beta \neq 0$.



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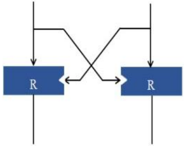
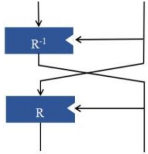


The closed butterfly V_R

- $\beta = (1+\alpha)^3$ and odd n:
  - Differential uniformity: 2^{n+1};
  - Non-linearity: $2^{2n-1} - 2^{(3n-1)/2}$;
  - Algebraic degree: n for H_R and 2 for V_R;
- $n = 3, \mathrm{Tr}(\alpha) = 0$ and $\beta \in \{\alpha^3 + \alpha, \alpha^3 + 1/\alpha\}$:
  - Differential uniformity: 2;
  - Non-linearity: $2^{2n-1} - 2^n$;
  - Algebraic degree: $n+1$ for H_R and 2 for V_R;
- Otherwise for odd n:
  - Differential uniformity: 4;
  - Non-linearity: $2^{2n-1} - 2^n$;
  - Algebraic degree: $n+1$ or n for H_R with $1 + \alpha\beta + \alpha^4 = (\beta + \alpha + \alpha^3)^2$ and 2 for V_R;

Previous generalizations

FSE 2018, Shihui Fu et. al.: $(x+\alpha y)^3 + y^3 \Rightarrow (x+\alpha y)^{2^i+1} + y^{2^i+1}$ with $\gcd(i,n) = 1$



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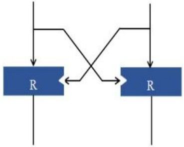

The closed butterfly V_R

- $\alpha \neq 0, 1$ and odd n:
  - Differential uniformity: at most 4;
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  - Non-linearity: $2^{2n-1} - 2^n$
- $\alpha = 1$ and odd n: $H_R \Leftrightarrow F^e$ and V_R is a permutation.
  - Differential uniformity: 4;
  - Non-linearity: $2^{2n-1} - 2^n$;
  - Algebraic degree: n for H_R and 2 for V_R;

$$
\begin{tikzcd}
 & (x+\alpha y)^3 + \beta y^3 \arrow[dr, dashed] & \\
(x+\alpha y)^3 + y^3 \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & (x+\alpha y)^{2^i+1} + \beta y^{2^i+1} \\
 & (x+\alpha y)^{2^i+1} + y^{2^i+1} \arrow[ur, dashed] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (x+\alpha y)^3+\beta y^3 \arrow[d, dotted] \\
(x+\alpha y)^3+y^3 \arrow[ur] \arrow[dr] \arrow[r, dotted] & (x+\alpha y)^{2^i+1}+\beta y^{2^i+1} \\
 & (x+\alpha y)^{2^i+1}+y^{2^i+1} \arrow[u, dotted]
\end{tikzcd}
$$

- How about the properties of more generalized butterflies?

$$(x+\alpha y)^e + y^e \Rightarrow (x+\alpha y)^e + \beta y^e$$

where $e = (2^i+1) \times 2^t$ with $\gcd(i,n) = 1$.

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$$(x+\alpha y)^e+y^e \Rightarrow (x+\alpha y)^e+\beta y^e$$

  where $e=(2^i+1)\times 2^t$ with $\gcd(i,n)=1$.

- The case of even n?

Our generalization and main result

$R(x,y) = (x+\alpha y)^e + \beta y^e$ where $e = (2^i + 1) \times 2^t$.
$(\alpha, \beta \neq 0$ with $\beta \neq (\alpha+1)^{2^i+1})$

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- odd n, $\gcd(i, n) = 1$:
  - Differential uniformity: at most 4;
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- odd n, $\gcd(i,n) = 1$:
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- $\gcd(i,n) = k$ and $\mathrm{Tr}\left(\frac{\beta}{\beta^2+(\alpha^2+1)^{2^i+1}}\right) = 1$ for $\mathrm{V_R}$:
    - Differential uniformity: at most 2^{2k};
    - Non-linearity: at least $2^{2n-1} - 2^{n+k_1-1}$, $k_1 = \gcd(2i,n)$;
    - Algebraic degree: 2.

Key point

Determine the number of solutions of a system of linear equations.

$$\begin{cases} a_1x^{2^i} + a_2x + b_1y^{2^i} + b_2y = c_1 \\ a_3x^{2^i} + a_4x + b_3y^{2^i} + b_4y = c_2. \end{cases}$$

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$$\Downarrow$$

Investigate the kernel of

$$L(x,y) = A \begin{pmatrix} x^{2^i} \\ x \end{pmatrix} + B \begin{pmatrix} y^{2^i} \\ y \end{pmatrix}$$

with $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

Relative results

Theorem 1

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ *be two nonzero matrices over* $\mathbb{F}_{2^n}$*, and* i *be an integer with* $\gcd(i, n) = k$*. Let*

$$L(x, y) = A \begin{pmatrix} x^{2^i} \\ x \end{pmatrix} + B \begin{pmatrix} y^{2^i} \\ y \end{pmatrix}$$

be a linear mapping from $\mathbb{F}_{2^n}^2$ *to* $\mathbb{F}_{2^n}^2$*. Then,* $|\ker(L(x, y))| \leq 2^{2k} \Leftrightarrow$

1. *When* $\mathrm{rank}(A) = 1$, $\mathrm{rank}\left(\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \end{pmatrix}\right) = 2$.
2. *When* $\mathrm{rank}(A) = 2$*, there does not exist* $\lambda \in \mathbb{F}_{2^n}^*$*, such that*

$$\begin{pmatrix} a_1\lambda^{2^i} & a_2\lambda \\ a_3\lambda^{2^i} & a_4\lambda \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Relative results

Lemma 1

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ *be two matrices over* $\mathbb{F}_{2^n}$*, and* i *be an integer with* $\gcd(i, n) = k$*. Let*

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be a linear mapping from $\mathbb{F}_{2^n}^2$ *to* $\mathbb{F}_{2^n}^2$*. If*

$$(a_1b_3 + a_3b_1) \neq 0 \text{ or } (a_2b_4 + a_4b_2) \neq 0,$$

$|\ker(L(x, y))| \leq 2^{2k}$.

Remark: Lemma 1 can be used to reduce the proof of the non-linearity of functions generated by 3-round Feistel network. ($\alpha = 1$)

An application of Lemma 1

n is odd, $\gcd(i,n)=1$ and $(a,c)\neq(0,0)\in\mathbb{F}_{2^n}^2$,

$$\begin{cases} a^{2^i}x^{2^{2i}}+ax+c^{2^i}y^{2^{2i}}+cy=0 \\ c^{2^i}x^{2^{2i}}+cx+(a+c)^{2^i}y^{2^{2i}}+(a+c)y=0. \end{cases}$$

has at most 4 solutions.

- $\gcd(2i,n)=\gcd(i,n)=1$
- $a^{2^i}(a+c)^{2^i}+c^{2^{i+1}}=0 \Leftrightarrow a(a+c)+c^2=0$
- $a^2+ac+c^2=0$ can not hold: For any $c\in\mathbb{F}_{2^n}^*$,

$$z^2+cz+c=0 \Leftrightarrow (z/c)^2+(z/c)+1=0$$

  has no solutions since $\mathrm{Tr}(1)=1$.

Proof of differential uniformity

Prove the system of linear equations below has at most 4 solutions for any $(a, b) \neq (0, 0) \in \mathbb{F}_{2^n}^2$.

$$\begin{cases} R_{\alpha,\beta}^{2^i+1}(x, y) + R_{\alpha,\beta}^{2^i+1}(x + a, y + b) + R_{\alpha,\beta}^{2^i+1}(a, b) = 0 \\ R_{\alpha,\beta}^{2^i+1}(y, x) + R_{\alpha,\beta}^{2^i+1}(y + b, x + a) + R_{\alpha,\beta}^{2^i+1}(b, a) = 0 \end{cases}$$

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$$\Downarrow \gamma = \alpha^{2^i+1} + \beta$$

$$\begin{cases} (a + \alpha b)x^{2^i} + (a + \alpha b)^{2^i} x + (\alpha^{2^i} a + \gamma b)y^{2^i} + (\alpha a^{2^i} + \gamma b^{2^i})y = 0 \\ (\gamma a + \alpha^{2^i} b)x^{2^i} + (\gamma a^{2^i} + \alpha b^{2^i})x + (\alpha a + b)y^{2^i} + (\alpha a + b)^{2^i} y = 0 \end{cases}$$

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$$\Downarrow \gamma = \alpha^{2^i+1} + \beta$$

$$\begin{cases} (a+\alpha b)x^{2^i} + (a+\alpha b)^{2^i}x + (\alpha^{2^i}a+\gamma b)y^{2^i} + (\alpha a^{2^i} + \gamma b^{2^i})y = 0 \\ (\gamma a + \alpha^{2^i}b)x^{2^i} + (\gamma a^{2^i} + \alpha b^{2^i})x + (\alpha a + b)y^{2^i} + (\alpha a + b)^{2^i}y = 0 \end{cases}$$

Applying the Lemma 1, we need to prove

$$\begin{aligned} (a+\alpha b)(\alpha a + b) &= (\gamma a + \alpha^{2^i}b)(\alpha^{2^i}a + \gamma b) \\ (a+\alpha b)^{2^i}(\alpha a + b)^{2^i} &= (\gamma a^{2^i} + \alpha b^{2^i})(\alpha a^{2^i} + \gamma b^{2^i}) \end{aligned}$$

cannot hold simultaneously.

$$\Updownarrow$$

Proof of differential uniformity

$$(\gamma\alpha^{2^i} + \alpha)a^2 + (\gamma^2 + \alpha^{2^{i+1}} + \alpha^2 + 1)ab + (\gamma\alpha^{2^i} + \alpha)b^2 = 0$$
$$(\gamma\alpha + \alpha^{2^i})a^{2^{i+1}} + (\gamma^2 + \alpha^{2^{i+1}} + \alpha^2 + 1)a^{2^i}b^{2^i} + (\gamma\alpha + \alpha^{2^i})b^{2^{i+1}} = 0.$$

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$b = 0$: $\gamma\alpha^{2^i} + \alpha = \gamma\alpha + \alpha^{2^i} = 0$ can not hold ;
$b \neq 0$: set $y = a/b$, the corresponding equations have no common solutions.

Proof of differential uniformity

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$b = 0$: $\gamma\alpha^{2^i} + \alpha = \gamma\alpha + \alpha^{2^i} = 0$ can not hold ;
$b \neq 0$: set $y = a/b$, the corresponding equations have no common solutions.

Lemma 2

Let n be odd and i be an integer with $\gcd(i, n) = 1$, $\alpha, \beta \in \mathbb{F}_{2^n}^*$. Let $\gamma = \alpha^{2^i+1} + \beta$, $D = \gamma\alpha^{2^i} + \alpha$, $E = (\alpha+1)^{2^i+1} + \beta$, $F = \alpha^{2^i} + \gamma\alpha$. Suppose $E \neq 0$. Then the equations

$$Dx^2 + E^2x + D = 0 \quad \text{and} \quad Fx^{2^{i+1}} + E^2x^{2^i} + F = 0$$

do not have common solutions in $\mathbb{F}_{2^n}$.

Proof of Non-linearity

Prove that for $(a,b),(c,d) \in \mathbb{F}_{2^n}^2$ with $(a,b) \neq (0,0)$,

$$|\lambda_{\mathsf{V}}((c,d),(a,b))| \leq 2^{n+1}.$$

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(1) Compute $\lambda_{\mathsf{V}}((c,d),(a,b))$:

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(1) Compute $\lambda_{\mathsf{V}}((c,d),(a,b))$: Let $\gamma = \alpha^{2^i+1} + \beta$

$$\lambda_{\mathsf{V}}((c,d),(a,b)) = \sum_{x,y \in \mathbb{F}_{2^n}} (-1)^{f(x,y)} \leq \mathcal{L}(f).$$

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$$\lambda_{\mathsf{V}}((c,d),(a,b)) = \sum_{x,y \in \mathbb{F}_{2^n}} (-1)^{f(x,y)} \leq \mathcal{L}(f).$$

$$f(x,y) = \mathrm{Tr}(Ax^{2^i+1} + Bx^{2^i}y + Cxy^{2^i} + Dy^{2^i+1})$$

with

$$A = a + b\gamma, B = a\alpha + b\alpha^{2^i}, C = a\alpha^{2^i} + b\alpha, D = a\gamma + b.$$

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(2) Determine $\mathcal{L}(f)$:

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Lemma [Anne Canteaut, Sebastien Duval, Leo Perrin. TIT 2017]

Let f be a quadratic Boolean function of n variables. Let $\mathrm{LS}(f)$ denote the linear space of f, i.e.

$$\mathrm{LS}(f) = \{a \in \mathbb{F}_{2^n} : D_a f(x) = \varepsilon, \forall x \in \mathbb{F}_{2^n}\},$$

where $\varepsilon \in \{0,1\}$. Then, $s = \dim \mathrm{LS}(f)$ has the same parity as n and $\mathcal{L}(f) = 2^{\frac{n+s}{2}}$. Moreover, the Walsh coeficients of f take 2^{n-s} times the value $\pm 2^{\frac{n+s}{2}}$ and $(2^n - 2^{n-s})$ times the value 0.

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$$\begin{cases} A^{2^i}u^{2^{2i}} + Au + C^{2^i}v^{2^{2i}} + Bv = 0, \\ B^{2^i}u^{2^{2i}} + Cu + D^{2^i}v^{2^{2i}} + Dv = 0. \end{cases}$$

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- $\begin{pmatrix} A^{2^i} & A \\ B^{2^i} & C \end{pmatrix}$ and $\begin{pmatrix} C^{2^i} & B \\ D^{2^i} & D \end{pmatrix}$ are nonzero matrices.

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- $\begin{pmatrix} A^{2^i} & A \\ B^{2^i} & C \end{pmatrix}$ and $\begin{pmatrix} C^{2^i} & B \\ D^{2^i} & D \end{pmatrix}$ are nonzero matrices.
- Discuss the rank of $\begin{pmatrix} A^{2^i} & A \\ B^{2^i} & C \end{pmatrix}$ and $\begin{pmatrix} A^{2^i} & A & C^{2^i} & B \\ B^{2^i} & C & D^{2^i} & D \end{pmatrix}$

in cases according to Theorem 1.

Comparison

Compare the number of CCZ-equivalent classes of V_R from different butterflies over a certain field.

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- Choose parameters:
  - $n = 5$ (the smallest for comparison)
  - $i = 1, 2$ ($\mathsf{V}_{\alpha,\beta}^{2^i+1}$ is EA-equivalent to $\mathsf{V}_{\alpha,\beta^{2^{n-i}}}^{2^{n-i}+1}$)

Comparison

Compare the number of CCZ-equivalent classes of V_R from different butterflies over a certain field.

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  - $i = 1, 2$ ($\mathsf{V}_{\alpha,\beta}^{2^i+1}$ is EA-equivalent to $\mathsf{V}_{\alpha,\beta^{2^{n-i}}}^{2^{n-i}+1}$)
- Determine all the CCZ-equivalent classes of $\mathsf{V}_{\alpha,\beta}^{2^i+1}$
  - $S = \{\mathsf{V}_{\alpha,\beta}^{2^i+1} : \alpha, \beta \in \mathbb{F}_{2^5}^*, \beta \neq (\alpha+1)^{2^i+1}, i = 1, 2\}$;
  - Choose $h \in S$, $S_h = \{f \in S : \text{IsEquivalent}(\tilde{C}_f, \tilde{C}_h) \text{ eq true}\}$;
  - Store S_h and let $S := S \setminus S_h$;
  - Repeat until $S = \emptyset$.

Experimental results

CCZ-inequivalence functions/permutations over $\mathbb{F}_{2^5}^2$ constructed with butterfly structure:

$R(x,y)$	Represent elements	Number
$i=1,\ \beta=1$	$\alpha=1, g^{33}, g^{99}, g^{165}, g^{231}, g^{363}, g^{495}$	7
$i=2,\ \beta=1$	$\alpha=1, g^{33}, g^{99}, g^{165}, g^{363}, g^{495}$	6
$i=1,\ \beta\neq 1$	$(\alpha,\beta)=(1,g^{33}), (1,g^{165}), (g^{33},g^{33}),$ $(g^{33},g^{165}), (g^{33},g^{693}), (g^{33},g^{726})$	6
$i=2,\ \beta\neq 1$	$(\alpha,\beta)=(1,g^{33}), (1,g^{363}), (1,g^{495}),$ $(g^{33},g^{99}), (g^{33},g^{132}), (g^{33},g^{198}), (g^{99},g^{165})$	7

The case of $\gcd(i, n) = k$

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Theorem 2

Let n, i be integers with $\gcd(i, n) = k$, $\alpha, \beta \in \mathbb{F}_{2^n}^*$ and $\beta \neq (\alpha + 1)^{2^i+1}$. Let $R_{\alpha,\beta}^{2^i+1}(x, y) = (x + \alpha y)^{2^i+1} + \beta y^{2^i+1}$ and

$$\mathsf{V}_{\alpha,\beta}^{2^i+1}(x, y) = (R_{\alpha,\beta}^{2^i+1}(x, y), R_{\alpha,\beta}^{2^i+1}(y, x)).$$

If $\mathrm{Tr}\left(\frac{\beta}{\beta^2+(\alpha^2+1)^{2^i+1}}\right) = 1$, then the following statements hold.

1. *The differential uniformity of $\mathsf{V}_{\alpha,\beta}^{2^i+1}$ is at most 2^{2k}.*
2. *The nonlinearity of $\mathsf{V}_{\alpha,\beta}^{2^i+1}$ is at least $2^{2n-1} - 2^{n+k_1-1}$, where $k_1 = \gcd(2i, n)$.*

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Remark: we can get differentially 4-uniform functions $\mathsf{V}_{\alpha,\beta}^{2^i+1}$ over $\mathbb{F}_{2^n}^2$ for any even n with $\gcd(i, n) = 1$.

Future work

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- More APN permutations from the generalized butterflies?
  (A sufficient condition for $|\ker(L(x,y))| \leq 2^k$?)

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- Conditions that make $\mathsf{V}_{\alpha,\beta}^{2^i+1}$ a permutation?

  Permutations that are CCZ-equivalent to $\mathsf{V}_{\alpha,\beta}^{2^i+1}$?

Thank you!